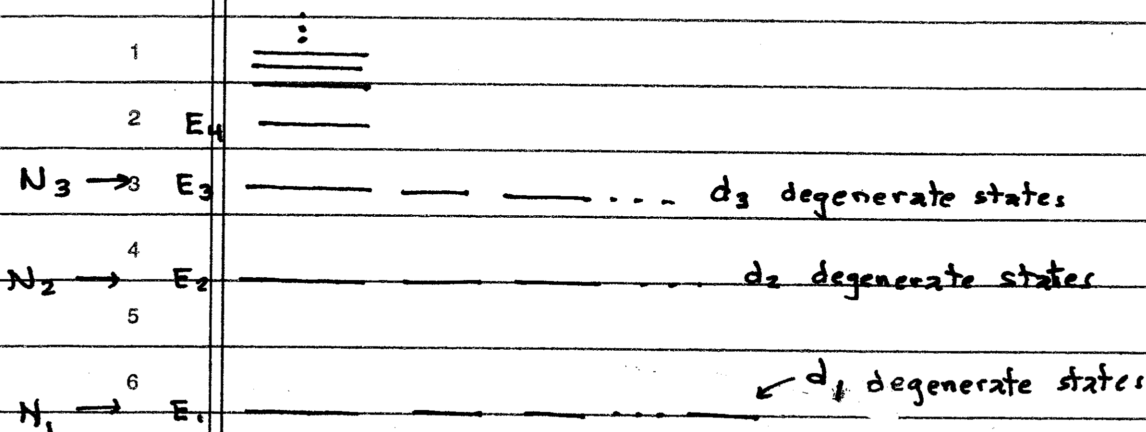
Prepared by:

Date:

Quantum Statistical Mechanics



$(N_1, N_2, N_3, \ldots) \rightarrow$ represents a particular configuration (macrostate)

$Q(N_1, N_2, N_3, \ldots) \rightarrow$ the number of microstates in that macrostate

or, "How many distinct states correspond to this particular configuration?"

① Distinguishable Particles:

$$\boxed{Q(N_1, N_2, N_3, \ldots) = N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}}$$

② Indistinguishable Particles: Fermions

$$Q(N_n) = \binom{d_n}{N_n} = \frac{d_n!}{N_n!(d_n - N_n)!} \quad \text{for the } n^{th} \text{ level}$$

$$\boxed{Q(N_1, N_2, N_3, \ldots) = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!(d_n - N_n)!}} \quad \text{Fermions}$$

③ Indistinguishable Particles: Bosons

$\bullet \; \bullet \; \times \; \bullet \; \times \; \bullet \; \bullet \; \bullet \; \times \; \bullet \; \times \qquad N_n = 7 \quad d_n = 5$

If the particles and partitions were "labeled," there would be $(N_n + d_n - 1)!$ different ways to arrange them. The dots and crosses are separately equivalent

$N_n!$ ways to arrange the dots $\rightarrow$ no new microstates

$(d_n - 1)!$ ways to arrange the crosses $\rightarrow$ no new microstates

So, there are $\dfrac{(N_n + d_n - 1)!}{N_n!\,(d_n - 1)!}$ ways of assigning N_n particles to the d_n degenerate states in the n^{th} bin.

Therefore,

$$\boxed{Q(N_1, N_2, N_3, \ldots) = \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!\,(d_n - 1)!}} \qquad \underline{\textbf{Bosons}}$$

<u>Most Probable Configuration</u>

In thermal equilibrium, every state with a given total energy E and a given number of particles N is <u>equally likely</u>.

The <u>most probable</u> configuration $(N_1, N_2, N_3, \ldots)$ is the one that can be achieved in the largest number of different ways.

It is the configuration where $Q(N_1, N_2, \ldots)$ is a maximum, subject to the constraint

$$\sum_{n=1}^{\infty} N_n = N \qquad \text{and} \qquad \sum_{n=0}^{\infty} N_n E_n = E$$

If $Q(N_1, N_2, N_3, \ldots)$ is a maximum, so is $\ln Q(N_1, N_2, N_3, \ldots)$

Introduce 2 Lagrangian multipliers for the 2 constraints $(\overset{N}{\alpha}, \overset{E}{\beta})$ and create an auxiliary function to maximize.

$$G(N_1, N_2, N_3, \ldots \overset{\alpha,\beta}{}) = \ln Q(N_1, N_2, N_3, \ldots) + \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right]$$

1 <u>Distinguishable:</u>

$$G(N_1, N_2, N_3, \ldots, \alpha, \beta) = \ln(N!) + \sum_{n=1}^{\infty}\left[N_n \ln d_n - \ln(N_n!)\right] +$$

$$+ \alpha\left[N - \sum N_n\right] + \beta\left[E - \sum N_n E_n\right]$$

Introduce Sterling's approximation $\ln z! = z \ln z - z$ for $z \gg 1$

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Distinguishable Particles

$$G(N_1, N_2, \ldots, \alpha, \beta) \approx \sum_{n=1}^{\infty} \left[N_n \ln d_n - N_n \ln N_n + N_n - \alpha N_n - \beta N_n E_n \right]$$

$$+ \ln(N!) + \alpha N + \beta E$$

Maximize $G(N_1, N_2, \ldots, \alpha, \beta)$

$$\frac{\partial G}{\partial N_n} = \ln(d_n) - \ln(N_n) - \alpha - \beta E_n \quad = 0$$

Solving for N_n we find: $\boxed{N_n = d_n e^{-(\alpha + \beta E_n)}}$ Distinguishable Particles

Identical Fermions:

$$G(N_1, N_2, \ldots, \alpha, \beta) = \sum_{n=1}^{\infty} \left\{ \ln(d_n!) - \ln(N_n!) - \ln\left[(d_n - N_n)!\right] \right.$$

$$\left. + \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right] \right.$$

Stirling's Approximation

$$G(N_1, N_2, \ldots, \alpha, \beta) \cong \sum_{n=1}^{\infty} \left[\ln(d_n!) - N_n \ln(N_n) + N_n - (d_n - N_n)\ln(d_n - N_n) \right.$$

$$\left. + (d_n - N_n) - \alpha N_n - \beta E_n N_n \right] + \alpha N + \beta E$$

$$\frac{\partial G}{\partial N_N} = -\ln N_n + \ln\left(d_n - N_n\right) - \alpha - \beta E_n \quad = 0$$

$$\boxed{N_n = \frac{d_n}{e^{(\alpha + \beta E_n)} + 1}}$$ Fermions

Identical Bosons: Following the same procedure as above with Lagrangian multipliers:

$$\boxed{N_n = \frac{d_n - 1}{e^{(\alpha + \beta E_n)} - 1}}$$ Bosons

Substitute the occupation numbers, N_n, into the constraint equations $\rightarrow$ $N = \sum_{n=1}^{\infty} N_n$ and $E = \sum_{n=1}^{\infty} N_n E_n$

To carry out the sums, we need to know the allowed energies (E_n) and their degeneracies (d_n) for the potential in question.

<u>Ideal Gas</u> $\rightarrow$ $E_k = \dfrac{\hbar^2 k^2}{2m}$ $\qquad d_k \rightarrow \dfrac{1}{8}\,\dfrac{\overset{\text{Thin spherical shell}}{4\pi k^2\,dk}}{\underset{\text{volume of a cell in k-space}}{\left(\pi^3/V\right)}} = \dfrac{V}{2\pi^2}\,k^2\,dk$

The 1^{st} constraint for distinguishable particles is $\sum_{n=1}^{\infty} N_n = N$

$\rightarrow N = \int dN_k \quad dN_k = d_k e^{-(\alpha+\beta E_k)}$

$$N = \frac{V}{2\pi^2} e^{-\alpha} \int_0^{\infty} e^{-\beta E_k} k^2\,dk$$

$$\rightarrow e^{-\alpha} = \frac{N}{V}\left(\frac{2\pi\beta\hbar^2}{m}\right)^{3/2}$$

The 2^{nd} constraint for distinguishable particles is $E = \sum_{n=1}^{\infty} N_n E_n$

or $E = \int dN_k\, E_k \Rightarrow E = \dfrac{V}{2\pi^2} e^{-\alpha} \dfrac{\hbar^2}{2m} \displaystyle\int_0^{\infty} e^{-\frac{\beta\hbar^2 k^2}{2m}} k^4\,dk$

Substituting the $e^{-\alpha}$ from up above we obtain: $\boxed{E = \dfrac{3N}{2\beta}}$

correct for all spins

E = total energy of an ideal gas for N particles:

$$\boxed{\beta \equiv \frac{1}{k_B T}}$$

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Replace α with the chemical potential $\mu(T)$ $\boxed{\mu(T) = -\alpha k_B T}$

To obtain the most probable number of particles in a particular state with energy ϵ, we simply divide by the degeneracy.

$$N_n(\epsilon_n) \longrightarrow n(\epsilon)$$

\# of particles with a given energy $\longrightarrow$ \# of particles in a particular (one particle) state with energy ϵ

$$n(\epsilon) = e^{-(\epsilon-\mu)/k_B T} \qquad \text{Maxwell-Boltzmann}$$

$$n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} + 1} \qquad \text{Fermi-Dirac}$$

$$n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1} \qquad \text{Bose-Einstein}$$

Fermi-Dirac Distribution at $T \to 0$

$$e^{(\epsilon-\mu)/k_B T} \to \begin{cases} 0 & \text{if } \epsilon < \mu(0) \\ \infty & \text{if } \epsilon > \mu(0) \end{cases}$$

So,

$$n(\epsilon) \to \begin{cases} 1 & \text{if } \epsilon < \mu(0) \\ 0 & \text{if } \epsilon > \mu(0) \end{cases}$$

$\mu(0) = E_F$ the Fermi energy

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Ideal Gas for Distinguishable Particles

1.) The total energy for N particles at temperature T is:

$$\boxed{E = \frac{3}{2} N k_B T}$$

2.) The chemical potential is:

Recall: $\mu(T) = -\alpha k_B T \qquad e^{-\alpha} = \frac{N}{V}\left(\frac{2\pi\beta\hbar^2}{m}\right)^{3/2}$

$$\boxed{\mu(T) = k_B T\left[\ln\frac{N}{V} + \frac{3}{2}\ln\left(\frac{2\pi\hbar^2}{m k_B T}\right)\right]}$$

What are N and E for Identical Particles?

$$N = \frac{V}{2\pi^2}\int_0^\infty \frac{k^2}{e^{[\hbar^2k^2/2m-\mu]/k_BT} \pm 1}\,dk$$

(−) Bosons

(+) Fermions

⇐ get $\mu(T)$ for a given $\frac{N}{V}$

$$E = \frac{V}{2\pi^2}\frac{\hbar^2}{2m}\int_0^\infty \frac{k^4}{e^{[\hbar^2k^2/2m-\mu]/k_BT} \pm 1}\,dk$$

⇐ get $E(T)$

Heat Capacity $= C = \frac{\partial E}{\partial T}$

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The Blackbody Spectrum Photons Spin Ang. Mom = $1\hbar$ $m_\ell = +1$ or -1

Massless particles

$E = \frac{\hbar^2 k^2}{2m} \rightarrow \hbar\omega$ (massless particle) $k \equiv \frac{2\pi}{\lambda} = \frac{\omega}{c}$

The # of photons is not a conserved quantity. $\frac{N}{V}$ increases with T.
Therefore, the constraint $N = \sum N_n$ does not apply.
Therefore, we set $\alpha = 0$ (the Lagrangian multiplier $\rightarrow \mu(T) \overset{\circ}{=}$)

$n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1}$ However $\mu \rightarrow 0$ so, $n(\epsilon) = \frac{1}{e^{\epsilon/k_B T} - 1}$

The degeneracy d_k comes from the calculation we did for an Ideal Gas.

$d_k = \frac{V}{2\pi^2} k^2\, dk$ $\times 2$ for 2 spin states and substitute $k \rightarrow \frac{\omega}{c}$

$d_\omega = \frac{V}{\pi^2} \frac{\omega^2}{c^3} d\omega$ $\Leftarrow$ Ultraviolet catastrophe

$N(\omega) = \frac{d_\omega}{e^{\hbar\omega/k_B T} - 1}$ $\Leftarrow$ Saves us (theoretically) from the ultraviolet catastrophe.

Energy Density $\rightarrow$
unit ang. frequency

$$\boxed{\rho(\omega) = \frac{N(\omega)\,\hbar\omega}{V} = \frac{\hbar\omega^3}{\pi^2 c^3 \left(e^{\hbar\omega/k_B T} - 1\right)}}$$

The energy/volume in an interval $d\omega$ is just $\rho(\omega)\, d\omega$

Problem 5.30 : a.) find $\rho(\lambda)$
b.) find λ_{max} $\rightarrow$ $\lambda_{max} = \frac{2.90 \times 10^{-3}\ \text{m}\cdot\text{K}}{T}$

Problem 5.31 Show that $\frac{E}{V} = \left(\frac{\pi^2 k_B^4}{15\, \hbar^3 c^3}\right) T^4$

How can you find σ? $\frac{P}{A} = \sigma T^4$

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Prob. 5.35 White Dwarf Stars

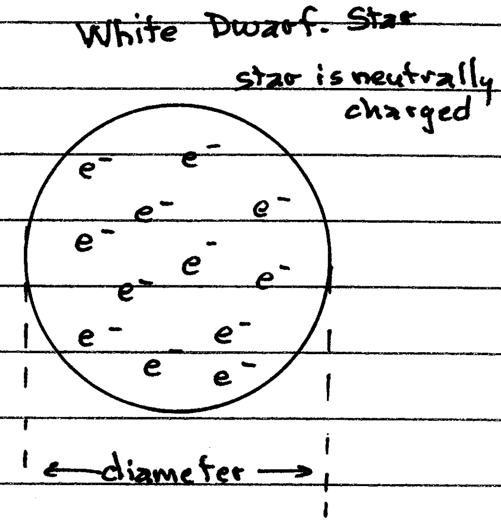
From electrons in a solid, we found

Eq. 5.45 $E_{TOT} = \dfrac{\hbar^2 (3\pi^2 Nq)^{5/3}}{10\pi^2 m} V^{-2/3}$

and this exerts a pressure $P = \dfrac{2}{3}\dfrac{E_{TOT}}{V} = \dfrac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3}$

where $\rho = \dfrac{Nq}{V}$

Electron degeneracy pressure due to Pauli Exclusion Principle. Similar to electrons in solids.



c.) Find where $E = KE + PE$ is a min.

$KE = E_{TOT}$ (see above)

$PE_{gravity} = -\dfrac{3}{5} G \dfrac{N^2 M^2}{R}$

$m \to$ mass of the electron $M \to$ mass of a proton

Radius $R \to 7.6 \times 10^{25}\ N^{-1/3}$ (meters)

e.) Determine the Fermi energy (eV) and compare it to the rest energy of an e^-.

What is the γ factor? $\gamma = \dfrac{E}{m_0 c^2} = \dfrac{KE + m_0 c^2}{m_0 c^2} = \dfrac{\overset{E_F}{KE}}{m_0 c^2} + 1$

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Prob. 5.36 Neutron Star.

a.) Extend the free electron gas (previous problem) to the relativistic domain. $\underbrace{KE = \frac{p^2}{2m}}_{\text{classical}} \rightarrow \underbrace{(\gamma - 1) m_0 c^2}_{\text{relativistic}}$

Relativistic Quantities: $\vec{p} = \hbar \vec{k}$ $\quad E \cong pc = \hbar k c$

Recalculate E_{TOT}:

Rewrite Eq. 5.44 (classical) $\quad dE = \underbrace{\frac{\hbar^2 k^2}{2m}}_{\text{classical}} \frac{V}{\pi^2} k^2 dk$

$dE = \hbar k c \frac{V}{\pi^2} k^2 dk$ and find E_{TOT}.

b.) Repeat parts (a.) and (b.) in prob. 5.35 for the ultrarelativistic electron gas.

$\Rightarrow$ $E = KE + PE$ has no stable minimum as a function of R

Find the critical number of nucleons, N_c, such that gravitational collapse occurs for $N > N_c$.

$N_c = 2.04 \times 10^{57}$ (Chandrasekhar Limit) $\quad E_{TOT} = \underbrace{\frac{A}{R}}_{KE} - \underbrace{\frac{B}{R}}_{PE}$

Stars more massive than this $\rightarrow$ gravitational collapse

$\rightarrow$ neutron star

$e^- p \rightarrow n \nu_e$

$\Rightarrow$ Now, you have degenerate neutron pressure. Pauli Exclusion Principle.

c.) Radius of the neutron star. $m \rightarrow M$ $\quad q \rightarrow 1$

$\Rightarrow$ $R = 12.4$ km

$\Rightarrow$ $E_F = 56.0$ MeV $\quad m_n c^2 = 940$ MeV for a neutron